# A Shortcut from Categorical Quantum Theory to Convex Operational Theories*

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**Abstract**

This paper charts a very direct path between the categorical approach to quantum mechanics, due to Abramsky and Coecke, and the older convex-operational approach based on ordered vector spaces (recently reincarnated as "generalized probabilistic theories"). In the former, the objects of a symmetric monoidal category $\mathcal{C}$ are understood to represent physical systems and morphisms, physical processes. Elements of the monoid $\mathcal{C}(I,I)$ are interpreted somewhat metaphorically as probabilities. Any monoid homomorphism from the scalars of a symmetric monoidal category $\mathcal{C}$ gives rise to a covariant functor $V_o$ from $\mathcal{C}$ to a category of dual-pairs of ordered vector spaces. Specifying a natural transformation $u : V_o \to 1$ (where 1 is the trivial such functor) allows us to identify normalized states, and, thus, to regard the image category $V_o(\mathcal{C})$ as consisting of concrete operational models. In this case, if $A$ and $B$ are objects in $\mathcal{C}$, then $V_o(A \otimes B)$ defines a non-signaling composite of $V_o(A)$ and $V_o(B)$. Provided either that $\mathcal{C}$ satisfies a "local tomography" condition, or that $\mathcal{C}$ is compact closed, this defines a symmetric monoidal structure on the image category, and makes $V_o$ a (strict) monoidal functor.

## 1 Introduction

In the categorical quantum mechanics of Abramsky and Coecke [2], physical theories are understood as symmetric monoidal categories, with physical systems as objects, physical processes as morphisms, and the monoidal structure allowing for the composition of systems and processes "in parallel". The scalars in such a category play the role, in a somewhat metaphorical sense, of probabilities. An older tradition, going back at least to the work of Ludwig [18], Davies and Lewis [12] and Edwards [13], models a physical system more concretely in terms of a dual pair of ordered vector spaces, one spanned by the system's states, the other by "effects" (essentially, measurement outcomes), with the duality prescribing the probability with which any given effect will occur in any given state. These concrete "operational" models can be combined by means of various possible non-signaling products [6].

Several attempts have been made to connect the two approaches. On one hand, [21, 11, 10, 15] have considered categories equipped with additional structure mirroring some of the structure found in the more concrete approach. On another hand, symmetric monoidal categories of concrete operational models have been constructed and studied in, e.g., [7, 22]. This paper attempts to link the categorical and operational approaches in a much more direct way, by constructing *representations* of (essentially arbitrary) symmetric monoidal categories *as* monoidal categories of concrete probabilistic models. The basic idea is simply to posit an homomorphism $p : \mathcal{C}(I,I) \to \mathbb{R}_+$ from the commutative monoid of scalars of the category, to the multiplicative monoid of non-negative real numbers, providing an interpretation of (some) scalars as genuine probabilities.

Depending on the model category $\mathcal{C}$ one has in mind, the set $\mathcal{C}(I,A)$ may best be taken to represent the pure states of system $A$, or the set of mixed states, or possibly a set of sub-normalized — or even, totally un-normalized — states. So far as possible, one would like to be able to deal with all of these cases in a reasonably uniform manner. To this end, I first construct, for a given $p$, a (more or less obvious)

*This is a revised and expanded version of notes privately circulated around 2010. The results on compact closure are new.

functor $V_o : \mathcal{C} \to \mathbf{OrdLin}$ from $\mathcal{C}$ into the category of ordered real vector spaces and positive linear mappings. There is a well-defined bilinear product $V_o(A), V_o(B) \mapsto V_o(A \otimes B)$ on the image category $V_o(\mathcal{C})$, satisfying certain desiderata for a composite of convex operational models (Proposition 4.1). Under an additional local tomography assumption (satisfied by all of the usual examples, but which one would certainly like to weaken), *or* if $\mathcal{C}$ is compact closed, this product makes $V_o(\mathcal{C})$ monoidal, and $V_o$, a strict monoidal functor (Propositions 4.2 and 4.3).

One can also define, for each $A \in \mathcal{C}$, an ordered space $V_o^{\#}(A) \leq V_o(A)^*$ spanned by the evaluation functionals associated with elements of $\mathcal{C}(A, I)$. By of construction, this is separating.

This much depends only on the monoid homomorphism $p$. To distinguish between normalized and non-normalized states, a bit more is required, namely, for each object $A \in \mathcal{C}$, a posited *unit effect* $u_A \in V_o(A)^*$. This is meant to represents the trivial event that is certain to occur. Accordingly, one *defines* a normalized state to be an element of $\alpha \in V_o(A)_+$ with $u_A(\alpha) = 1$. Such states form a convex set $\Omega_o(A)$, which is a base for the positive cone $V_o(A)_+$ (that is, every element of the cone is a multiple of normalized state).

We will actually require a little more still. First, in order for normalization to behave correctly under the composition, we will ask that $u_{A\otimes B} = u_A \otimes u_B$ for all $A, B \in \mathcal{C}$. In order to allow us to interpret elements of $\mathcal{C}(A, I)$ as being at least multiples of effects, we also require that, for every $a \in \mathcal{C}(A, I)$, the corresponding evaluation functional on $V_o(A)$ be dominated by a positive multiple of $u$, i.e,

$$\forall a \in \mathcal{C}(A, I)\ \exists t \geq 0\ a \leq t u_A. \tag{1}$$

Let $V_o^{\#}(A)$ denote the (by construction, separating) ordered subspace of $V_o(A)^*$ spanned by the evaluation functionals associated with elements of $\mathcal{C}(A, I)$, The unit effect $u_A$ will often belong to $V_o^{\#}$; when it does, (1) makes it an order unit for $V_o^{\#}(A)$, and the triple $(V_o(A), V_o^{\#}(A), u_A)$ is then a convex operational model as defined, e.g., in [7].

One can also consider the pointwise closure, $\Omega(A)$, of $\Omega_o(A)$ in $\mathbb{R}^{\mathcal{C}(A,I)}$: this is a compact convex set, and thus generates a complete base-normed space $V(A)$ (coinciding with $V_o$ when $V_o(A)$ is finite-dimensional). Since the choice of order-unit $u$ is not canonical, $V$ is not functorial on $\mathcal{C}$. are the object parts of functors $\mathcal{C}_u \to \mathbf{OrdLin}$, and we *conjecture* that $(V(A\otimes B), V^{\#}(A\otimes B), u_{AB})$ is a "non-signaling" product of $(V(A), V^{\#}(A), u_A)$ and $(V(B), V^{\#}(B), u_B)$.

These constructions are illustrated for a number of model categories $\mathcal{C}$, including the categories of finite-dimensional Hilbert spaces (where we recover the expected thing) and the category of finite sets and binary relations. Although I have tried to leave ample room for infinite-dimensional examples, I've avoided the head-on engagement with the linear-topological issues that this project will ultimately require.

# 2 Preliminaries

I denote the category of (all!) real vector spaces and linear mappings by **RVec**; however, I write $\mathcal{L}(V, W)$, rather than $\mathbf{RVec}(V, W)$, for the space of linear mappings $V \to W$, $V, W \in \mathbf{RVec}$. I write $V^*$ for the full (algebraic) dual of $V$, $V \otimes W$ for the algebraic tensor product of $V$ and $W$, and $\mathcal{B}(V, W)$ for the space of bilinear forms $V \times W \to \mathbb{R}$. By an *ordered vector space*, I mean a real vector space $V$ equipped with a convex, pointed, generating cone $V_+$. Any space of the form $\mathbb{R}^X$, $X$ a set, will be understood to be ordered pointwise on $X$. If $V, W$ are ordered linear spaces, a mapping $f : V \to W$ is *positive* iff $f(V_+) \subseteq W_+$. The *dual cone* of an ordered linear space $V$ is the cone $V_+^*$ consisting of all positive linear functionals on $V$ (where $\mathbb{R}$ is understood to have its usual order). The span of $V_+^*$ in $V^*$ is called the *order dual* of $V$, and denoted $V^\star$ I write **OrdLin** for the category of ordered linear spaces and positive linear mappings.

**Representations.** A (real, linear) *representation* of a category $\mathcal{C}$ is simply a covariant functor $V : \mathcal{C} \to \mathbf{RVec}$. There are two standard functors $\mathbf{Set} \to \mathbf{RVec}$, one contravariant, given on objects by $X \mapsto \mathbb{R}^X$, and the other covariant, given on objects by $X \mapsto \mathbb{R}^{[X]}$, where the latter is the vector space generated by $X$, or, equivalently, the space of finitely non-zero functions on $X$. Thus, given a reference object $I \in \mathcal{C}$, we have two basic representations $\mathcal{C} \to \mathbf{RVec}$, given on objects by $A \mapsto \mathbb{R}^{\mathcal{C}(A,I)}$ and by $A \mapsto \mathbb{R}^{[\mathcal{C}(I,A)]}$. A representation $V$ is *finite-dimensional* iff $V(A)$ is finite-dimensional for every object $A \in \mathcal{C}$. By a representation of a *symmetric monoidal* category $\mathcal{C}$, I mean a functor $V : \mathcal{C} \to \mathbf{RVec}$ that is symmetric monoidal with respect to *some* well-defined bilinear product on the image category. Of course, given the functor, there is only one candidate for this product. The following is obvious, but worth stating explicitly.

**Lemma 2.1** *Let $V : \mathcal{C} \to \mathbf{RVec}$ be a functor such that the operations (i) $V(A), V(B) \to V(A \otimes B)$ and (ii) $V(\phi), V(\psi) \to V(\phi \otimes \psi)$ are well-defined*[1] *for $A, B \in \mathcal{C}$ and $\phi \in \mathcal{C}(A, C)$, $\psi \in \mathcal{C}(B, D)$. Then the image category $V(\mathcal{C})$ is monoidal with respect to the product given by $V(A) \circledast V(B) := V(A \otimes B)$ (with associators, left and right units, and swap morphisms carried over from $\mathcal{C}$). With respect to this structure, $V$ is a strict monoidal functor.*[2]

Notice that this says nothing about the bilinearity of the monoidal structure on $V(\mathcal{C})$. I return to this point below.

**Dual pairs and Convex Operational Models.** For present purposes, we may define a *dual pair of ordered vector spaces* — an *ordered dual pair*, for short — as a pair $(V, V^{\#})$, where $V$ is an ordered vector space and $V^{\#}$ is a subspace of $V^*$, ordered by a cone $V^{\#}_+$ contained in the dual cone $V^*_+$. In other words, if $b \in V^*_+$ and $\alpha \in V_+$, $b(\alpha) \geq 0$ (but it may be that $b(\alpha) \geq 0$ for all $\alpha \in V_+$, yet $b \notin V^{\#}_+$). I will also assume, without further comment, that $V^{\#}$ is separating, i.e., that if $\alpha \in V$ and $b(\alpha) = 0$ for all $b \in V^{\#}$, then $\alpha = 0$. The following language is borrowed from [7], but the idea is essentially the same one proposed by Ludwig [18], Davies and Lewis [12], Edwards [13] and others in the 1960s and 1970s as a general framework for post-classical probabilistic physics.

**Definition 2.2** A *convex operational model* (COM) is a triple $(V, V^{\#}, u)$ where $(V, V^{\#})$ is an ordered dual pair and $u \in V^{\#}$ is a chosen order unit.[3]

A COM gives us a very general environment in which to discuss probabilistic concepts. An element $\alpha$ of $V_+$ with $u(\alpha) = 1$ is a *normalized state* of the model. An *effect* of the model is an element $a$ of $V^{\#}_+$ with $a \leq u$; equivalently, $a(\alpha) \leq 1$ for all normalized states $\alpha$. Effects represent (mathematically) possible measurement outcomes: if $a$ is an effect and $\alpha$ is a normalized state, $a(\alpha)$ is interpreted as the *probability* that $a$ will occur (if measured) in state $\alpha$.

**Example 2.3 (Motivating Examples)** (a) Let $(S, \Sigma)$ be a measure space. Let $V = M(S, \Sigma)$, the space of all countably-additive measures on $S$, and $V^{\#} = B(S, \Sigma)$, the space of all bounded measurable functions on $S$, with the duality given by $f(\mu) := \int_S f(s) d\mu(s)$ for every $f \in B(S, \Sigma)$ and $\mu \in M(S, \Sigma)$. The constant function 1 serves as the order unit. (b) Let $\mathcal{H}$ be any Hilbert space: take $V = \mathcal{L}_1(\mathcal{H})$, the space of trace-class self-adjoint operators on $\mathcal{H}$, and $V^{\#} = \mathcal{L}_{\text{sa}}(\mathcal{H})$, the space of all bounded Hermitian operators on $\mathcal{H}$, with the duality given by the trace, i.e, if $\alpha$ is a trace-class operator and $a$, any bounded operator, $\sigma(\alpha, a) = \text{Tr}(a\alpha)$. The identity operator on $\mathcal{H}$ serves as the order unit.

**Composites** Suppose that $(V, V^{\#}, v)$ and $(W, W^{\#}, w)$ are two (convex operational) models, representing two physical systems. In attempting to form a reasonable model of a composite system , the most obvious construction — $(V \otimes W, V^{\#} \otimes W^{\#}, v \otimes w)$ — is rarely appropriate. Certainly in infinite dimensions, one will typically need to pass from $V \otimes W$ to some appropriate linear-topological completion; but even where $V$ and $W$ are finite-dimensional, there are at least two further issues:

- There is no one canonical choice for the cones $(V \otimes W)_+$ and $(V^{\#} \otimes W^{\#})_+$: there do exist minimal and maximal tensor cones [6], but in the quantum-mechanical examples, these yield the wrong things.
- As is well known, in the case of real or quaternionic quantum models, where $V = \mathcal{L}_{\text{sa}}(\mathcal{H})$ and $W = \mathcal{L}_{\text{sa}}(\mathcal{K})$ for finite-dimensional real or quaternionic Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, one finds, upon counting dimensions, that $\mathcal{L}_{\text{sa}}(\mathcal{H} \otimes \mathcal{K}) \not\simeq \mathcal{L}_{\text{sa}}(\mathcal{H}) \otimes \mathcal{L}_{\text{sa}}(\mathcal{K})$.

[1] That is to say: if $V(A) = V(A')$ and $V(B) = V(B')$, then $V(A \otimes B) = V(A' \otimes B')$, and similarly for morphisms.

[2] Henceforth, "monoidal" will always mean "strict monoidal".

[3] The so-called *no-restriction hypothesis* [16], usually stated informally as the requirement that all "mathematically possible" effects be physically realizable, amounts to the requirement that $V^{\#} = V^{\star}$, the order-dual of $V$. One wants to avoid this very strong assumption wherever possible.

How, then, *should* one define a composite of two models? At a minimum, one wants to be able to construct *joint measurements* and prepare the systems independently in any two states. One also wants to be able to form, from a joint state $\omega \in VW$, the *conditional* state of, say, $W$, given the result of a measurement (an effect) on the first system. This suggests the following definitions [22]. (A bilinear mapping is *positive* iff it takes positive values on positive arguments.)

**Definition 2.4** *A* product *of two ordered dual pairs $(V, V^{\#})$ and $(W, W^{\#})$ is an ordered dual pair $(VW, (VW)^{\#})$, together with (i) A positive bilinear mapping $\circledast : V \times W \to VW$, and (ii) A positive linear mapping $\Lambda : VW \to \mathcal{B}(V^{\#}, W^{\#})$ such that*

- *(a) $\Lambda(\alpha \circledast \beta)(a, b) = a(\alpha)b(\beta)$ for all $\alpha \in V, \beta \in W, a \in V^{\#}$ and $b \in W^{\#}$.*
- *(b) $\Lambda(\omega)(-, b) \in V_+$ and $\Lambda(\omega)(a, -) \in W_+$ for all $a \in V^{\#}_+$ and $b \in W^{\#}_+$*

Restricting the dual mapping $\Lambda^* : \mathcal{B}(V, W)^* \to (VW)^*$ to (the canonical image of) $V^{\#} \otimes W^{\#} \leq B(V, W)^*$, we have a linear mapping $\pi : V^{\#} \otimes W^{\#} \to (VW)^*$. Since $\pi(a \otimes b)(\alpha \circledast \beta) = a(\alpha)b(\beta)$, it will be convenient simply to write $a \otimes b$ for $\pi(a \otimes b)$.

**Definition 2.5** *A product of COMs $(V, V^{\#}, v)$ and $(W, W^{\#}, w)$ is a COM $(VW, (VW)^{\#}, vw)$ where $(VW, (VW)^{\#})$ is a product of ordered dual pairs, and $vw = v \otimes w$.*

If $a$ and $b$ are effects in $V^{\#}$ and $W^{\#}$, respectively, then $a \otimes b$ is an effect in $(VW)^{\#}$, called a *product effect*, and interpreted as the result of measuring $a$ and $b$ jointly on the systems represented by $(V, V^{\#}, u)$ and $(W, W^{\#}, v)$. We then have $\Lambda(\omega)(a, b) = (a \otimes b)(\omega)$. Accordingly, I refer to $\Lambda$ as the *localization mapping*. The idea is that if $\omega \in VW$ is a state of the composite system, then $\Lambda(\omega)$ is object assigning joint probabilities to pairs of outcomes of "local" measurements associated with the component systems, represented by $V$ and $W$, respectively. Anticipating later need, let us agree to call a product $(VW, (VW)^{\#}, vw)$ *locally tomographic* iff these local joint probabilities suffice to determine the state $\omega$: in other words, iff $\Lambda$ is injective.

*Remark:* The bilinearity of $\Lambda$ (or of $\pi$) is equivalent to the "no-signaling" condition. If $E = \{a_i\}$ is an observable of the COM $(V, V^{\#}, v)$, i.e, a set of effects summing to the unit $v$, and $\omega$ is a state of $\Omega(VW)$, then the *marginal state* of $B$, given this observable, is defined, $\forall b \in W^{\#}$, by $\omega_E(b) = \sum_i \Lambda(\omega)(a_i \otimes b) = \Lambda(\omega)(\sum_i a_i \otimes b) = \Lambda(\omega)(v \otimes b)$, which is evidently independent of $E$. The interpretation is that the probability of observing an effect $b$ on the system corresponding to $(W, W^{\#}, w)$ is independent of which measurement we make on the system corresponding to $(V, V^{\#}, v)$. This works equally well in the other direction. We thus have well-defined marginal states $\omega_1 = \Lambda(\omega)(v, \ \cdot \ )$ and $\omega_2 = \Lambda(\omega)( \ \cdot \ , w)$. Condition (ii)(b) guarantees that these actually belong to $W_+$ and to $V_+$, respectively, and not just to $(W^{\#})^*_+$ and $(V^{\#})^*_+$.

Given a functor $V : \mathcal{C} \to \mathbf{OrdLin}$ in which $V(I) = \mathbb{R}$, we have $V(a) \in V(A)^*$ for all $a \in \mathcal{C}(A, I)$. Letting $V^{\#}(A)$ denote the span, in $V(A)^*$, of the functionals $V(a)$, $a \in \mathcal{C}(A, I)$, we have a functor $A \mapsto (V(A), V^{\#}(A))$ from $\mathcal{C}$ to the category **OrDP** of real dual pairs. We should like this to be monoidal, in the sense that the obvious (and only) candidate for a monoidal product on the image category be well-defined, but also, yield products of ordered dual pairs, in the sense defined above, *and* interact with the monoidal structure carried over from that of $\mathcal{C}$ in a sensible way. The following definition attempts to make these requirements precise.

**Definition 2.6** A *monoidal ordered linear representation* of a symmetric monoidal category $\mathcal{C}$, is a functor $V : \mathcal{C} \to \mathbf{OrdLin}$, such that the constructions

$$V(A), V(B) \mapsto V(A \otimes B) \ \text{ and } \ V(\phi), V(\psi) \mapsto V(\phi \otimes \psi) : V(A \otimes C) \to W(B \otimes D)$$

with $\phi \in \mathcal{C}(A, B), \psi \in \mathcal{C}(C, D)$, are well-defined, together with, for all objects $A$ and $B$, (i) linear mappings $\circledast_{A,B} : V(A) \otimes V(B) \to V(A \otimes B)$ and (ii) linear mappings $\Lambda_{A,B} : V(A \otimes B) \to \mathcal{B}(V^{\#}(A), V^{\#}(B))$

making $(V(A \otimes B), V^{\#}(A \otimes B))$ a product in the sense of Definition 2.4, of $(V(A), V^{\#}(A))$ and $(V(B), V^{\#}(B))$, such that (iii)

$$V(\alpha) \circledast V(\beta) = V(\alpha \otimes \beta) \text{ and } \Lambda(V(\alpha \otimes \beta))(V(a), V(b)) = V(a \circ \alpha) V(b \circ \beta)$$

for all $\alpha \in \mathcal{C}(I, A)$, $a \in \mathcal{C}(A, I)$, $\beta \in \mathcal{C}(I, B)$, and $b \in \mathcal{C}(B, I)$.

*Remarks* (a) By Lemma 2.1, the image category $V(\mathcal{C})$ of $\mathcal{C}$ under such a representation will be a symmetric monoidal category, and $V$, a (strict) monoidal functor. Suppose now that $V(I) = \mathbb{R}$. Then every morphism $\alpha \in \mathcal{C}(I, A)$ gives rise to a vector $V(\alpha)(1) \in V(A)$. If $V(A)$ is spanned by vectors of this form, for all $A \in \mathcal{C}$, then condition (iii) is equivalent to the requirement that, for all $A, B, C, D \in \mathcal{C}$ and all morphisms $\phi : A \to C$ and $\psi : B \to D$, $V(\phi \otimes \psi)(v \circledast w) = V(\phi)(v) \circledast V(\psi)(w)$ for all vectors $v \in V(A)$ and $w \in V(B)$.

(b) Definition 2.6 could be stated more abstractly. Both of the mappings $(V \circledast V) : A, B \mapsto V(A \circledast B) := V(A \otimes B)$ and $(V \otimes V) : A, B \mapsto V(A) \otimes V(B)$ are (object parts of) functors from $\mathcal{C} \times \mathcal{C}$ into **RVec**. If we equip $V(A) \otimes V(B)$ with the minimal tensor cone, consisting of linear combinations $\sum_i t_i v_i \otimes w_i$ where $v_i \in V(A)_+$, $w_i \in V(B)_+$, and with coefficients $t_i \geq 0$, then a positive bilinear mapping $V(A) \times V(B) \to V(A \otimes B)$ extends uniquely to a positive linear mapping $V(A) \otimes V(B) \to V(A \otimes B)$. In this way, we can regard both $V \circledast V$ and $V \otimes V$ as functors $\mathcal{C} \times \mathcal{C} \to$ **OrdLin**. From this point of view, the family of positive bilinear mappings $\circledast_{A,B}$ posited in part (ii) are the components of a natural transformation $V \otimes V \stackrel{\cdot}{\to} V \circledast V$. Similarly, the we can regard the mappings $\Lambda_{A,B}$ as components of a natural transformation from $V \circledast V$ to $(V^{\#} \otimes V^{\#})^*$, where, again, $V^{\#}(A) \otimes V^{\#}(B)$ has the minimal tensor cone and its dual, the dual cone.

# 3 The Representation $V_o$

There is a particularly simple, and canonical, representation of any category $\mathcal{C}$ in **OrdLin**. As discussed above, there is a "largest" contravariant linearization functor **Set** $\to$ **RVec**, namely, $X \mapsto \mathbb{R}^X$, $f \mapsto f^*$, where, if $f : X \to Y$, $f^* : \mathbb{R}^Y \to \mathbb{R}^X$ is the linear mapping taking $\beta \in \mathbb{R}^Y$ to $f^*(\beta) = \beta \circ f$. Composing this with the contravariant **Set**-valued functor $A \mapsto \mathcal{C}(A, I)$, $\phi \mapsto \phi^*$, where, again, $\phi^*$ is defined by $\phi^*(a) = a \circ \phi$ for all $a \in \mathcal{C}(A, I)$, gives us a covariant functor $\mathcal{C} \to$ **RVec**, taking each object $A$ to the (huge) vector space $\mathbb{R}^{\mathcal{C}(A,I)}$ and each morphism $\phi \in \mathcal{C}(A, B)$ to the linear mapping $\phi_* : \mathbb{R}^{\mathcal{C}(A,I)} \to \mathbb{R}^{\mathcal{C}(B,I)}$ given by

$$\phi_*(\alpha)(b) = \alpha(\phi^*(b)) = \alpha(b \circ \phi)$$

for all $\alpha \in \mathbb{R}^{\mathcal{C}(A,I)}$ and all $b \in \mathcal{C}(B, I)$. (Of course, we can do the same using any vector space, or, for that matter, any set, in place of $\mathbb{R}$.) With respect to the natural pointwise ordering on spaces of the form $\mathbb{R}^X$, the linear mappings defined above are positive. Thus, we can regard the functor just defined as taking $\mathcal{C}$ to **OrdLin**, where the latter is the category of ordered linear spaces and positive linear mappings.

Suppose now that $\mathcal{C}$ is a symmetric monoidal category with tensor unit $I$. Let $S = \mathcal{C}(I, I)$ be the monoid of scalars in $\mathcal{C}$, and let $p : S \to \mathbb{R}_+$ be a monoid homomorphism (where we regard $\mathbb{R}_+$ as a monoid under multiplication). For each $\alpha \in \mathcal{C}(I, A)$, let $[\alpha] \in \mathbb{R}^{\mathcal{C}(A,I)}$ be the function defined by

$$[\alpha](a) = p(a \circ \alpha)$$

for all $a \in \mathcal{C}(A, I)$. Ultimately, we wish to be able to identify those $\alpha \in \mathcal{C}(I, A)$ and those $a \in \mathcal{C}(A, I)$ that correspond to actual physical states and effects (or events), and, for such a pair, to regard $[\alpha](a)$ as the *probability* that the effect $a$ occurs when the system $A$ is in state $\alpha$. This will require some further attention to questions of normalization, which we'll return to in section 5. Meanwhile, we are now in a position to represent elements of $\mathcal{C}(I, A)$ as elements of the positive cone of a ordered vector space:

**Definition 3.1** *Let $V_o(A)$ denote the linear span of the vectors $[\alpha] \in \mathbb{R}^{\mathcal{C}(A,I)}$, ordered pointwise.*[4]

[4]This notation suppresses the dependence of $V_o$ on the choice of monoid homomorphism $p$. Should it become necessary to track this dependence, we can write $V_o^p$ or something of the sort.

If $\phi \in \mathcal{C}(A, B)$, we have

$$\phi_*([\alpha])(b) = [\alpha](b \circ \phi) = p(b \circ \phi \circ \alpha) = [\phi \circ \alpha](b)$$

for all $b \in \mathcal{C}(B, I)$. Thus, we may regard $\phi_*$ as mapping $V_o(A)$ to $V_o(B)$. Writing $V_o(\phi)$ for $\phi_*$, thus restricted and co-restricted, we have a functor $V_o : \mathcal{C} \to \mathbf{OrdLin}$.

*Remark:* The functor $V_o$ will sometimes be degenerate. For instance, if $\mathcal{C}$ is a meet semi-lattice, with $a \otimes b = a \wedge b$ and $I = 1$ (the top element of $\mathcal{C}$), we have $\mathcal{C}(I, I) = \{1\}$, and there is a unique monoid homomorphism $p : S \to \mathbb{R}_+$. However, for $a \neq I$, $\mathcal{C}(I, a) = \emptyset$, whence, $V_o(a)$ is again empty.

**Lemma 3.2** *For any SMC $\mathcal{C}$, $V_o(I)$ is canonically isomorphic to $\mathbb{R}$.*

*Proof:* For all $s, t \in \mathcal{C}(I, I)$, we have $[s](t) = p(s \circ t) = p(s)p(t)$. Thus, $[s] = p(s)p \in \mathbb{R}^S = V(S)$, whence, $V_o(I)$ is the one-dimensional span of $p \in V_o(I)$. $\square$

Up to this canonical isomorphism $V_o(I) \simeq \mathbb{R}$, we can now identify $V_o(s)$ with $p(s)$ for all $s \in \mathcal{C}(I, I)$. Also, for $\alpha \in \mathcal{C}(I, A)$, we have $V_o(\alpha) : V_o(I) \to V_o(A)$. Up to the canonical isomorphism $V_o(I) \simeq \mathbb{R}$, we have $V_o(\alpha)(1) = [\alpha]$. Similarly, if $a \in \mathcal{C}(A, I)$, have $V_o(a) : V_o(A) \to V_o(I) \simeq \mathbb{R}$, i.e., $V_o(a) \in V_o(A)^*$, given by $V_o(a)(\rho)(1) = \rho(a)$. Denoting the evaluation functional $\rho \mapsto \rho(a)$ by $[a]$, we have $V_o(a) = [a]$ (again, up to the identification of $V_o(I)$ with $\mathbb{R}$). Note that $[a]([\alpha]) = [\alpha](a) = V_o(\alpha \circ a)$.

Note that the evaluation functional $[a] \in V_o(A)^*$ is positive for all $a \in \mathcal{C}(A, I)$.

**Definition 3.3** *Let $V_o^{\#}(A)$ denote the span of these functionals $[a]$, ordered by the cone they generate.*

Note that, for $\phi \in \mathcal{C}(A, B)$, we have a dual mapping $(V_o\phi)^{\#} : V_o^{\#}(B) \to V_o^{\#}(A)$, and hence, dual to this, a mapping $V_o(\phi) := (\phi^{\#})^* : V_o(A) \to V_o(B)$. Thus, the ordered dual pair $(V_o(A), V_o^{\#}(A))$ is functorial in $A$. In what follows I will live a bit dangerously and simply write $a$ for $[a] \in V_o^{\#}(A)$, leaving it to context to disambiguate usage. (In particular, I am *not* assuming that $a \mapsto [a]$ is injective.)

**Example 3.4 (Finite-dimensional Hilbert spaces)** Let **FdHilb** be the category of finite-dimensional complex Hilbert spaces and linear mappings. We have have $\mathcal{C}(I, A) \simeq \mathcal{C}(A, I) \simeq A$, where $\alpha : I \to A$ is determined by the vector $\alpha(1) \in A$, while $a \in \mathcal{C}(A, I)$ is determined by $a(x) = \langle v_a, x \rangle$ for a unique $v_a \in A$. The monoid $\mathcal{C}(A, A)$ is (isomorphic to) $(\mathbb{C}, \cdot)$. Let $p : \mathbb{C} \to \mathbb{R}_+$ be given by $p(z) = |z|^2$. If $\alpha \in \mathcal{C}(A, I)$ and $a \in \mathcal{C}(A, I)$ then $\alpha(a) = p(a \circ \alpha) = |\langle v_a, \alpha(1) \rangle|^2$, which gives the usual quantum-mechanical transition probability. From this point forward, we identify $a \in \mathcal{C}(A, I)$ with $v_a \in A$; then we may interpret $[\alpha] \in \mathbb{R}^A$, via $[\alpha](a) = |\langle a | \alpha(1) \rangle|^2 = \langle \alpha(1) \odot \alpha(1) a, a \rangle$ — in other words, $[\alpha]$ is the quadratic form associated with the rank-one operator $\alpha(1) \odot \alpha(1)$.[5] It follows that $V_o(A)$ is the space of (quadratic forms associated with) Hermitian operators on $A$. We also have $V_o^{\#}$ the span of rank-one Hermitian operators — in our finite-dimensional setting, then, $V_o^{\#}(A) \simeq V_o(A)$.

If $\phi \in \mathcal{C}(A, B) = \mathcal{L}(A, B)$, we have $v_{b \circ \phi} = v_{\phi^*(b)}$, i.e., $b \circ \phi = \phi^*(b)$. Hence,

$$V_o(\phi)([\alpha])(b) = [\alpha](b \circ \phi) = \langle (\alpha(1) \odot \alpha(1))\phi^*(b), \phi^*(b) \rangle = \langle \phi(\alpha(1) \odot \alpha(1))\phi^* b, b \rangle,$$

so that $V_o(\phi)([\alpha]) = [\phi(\alpha)]$; note that this also shows that $V_o(\phi) : \rho \mapsto \phi \rho \phi^*$ for all $\rho \in V_o(A) \simeq \mathcal{L}_h(A)$, i.e., $V$ implements the usual lifting of linear mappings from $A$ to $B$ to linear mappings from $\mathcal{L}_h(A)$ to $\mathcal{L}_h(B)$.

**Example 3.5 (Arbitrary Hilbert spaces)** Let **Hilb** the category of all separable complex Hilbert spaces, and bounded linear mappings. Again, we have $\mathcal{C}(A, I) \simeq A$. Here $V_o(A)$ is the space of finite-rank Hermitian operators on $A$. space of finite-rank operators. The same computation as above shows that, for a bounded linear mapping $\phi : A \to B$, $V_o\phi : V_o(A) \to V_o(B)$ is the conjugation mapping $\rho \mapsto \phi \rho \phi^*$.

[5] Here, $a \odot b : \mathcal{H} \to \mathcal{H}$ is given by $(a \odot b)x = \langle x, b \rangle a$.

**Example 3.6 (Relations)** Let $\mathcal{C} = \mathbf{Rel}$, the category of sets and relations. The tensor unit is the one-point set $I = \{*\}$, so that $S = \{\emptyset, \{(*,*)\}\} \simeq \mathcal{P}(I \times I) \simeq \mathcal{P}(I)$. Let's identify this with $\{0,1\} \subseteq \mathbb{R}_+$. We also have, for every $A \in \mathcal{C}$, isomorphisms $\mathcal{C}(A,I) \simeq \mathcal{C}(I,A) \simeq \mathcal{P}(A)$, with $\alpha \in \mathcal{C}(I,A)$ corresponding to $\alpha(*) \subseteq A$ and $a \in \mathcal{C}(A,I)$, to $a^{-1}(*) \subseteq A$. Let $p : S = \{0,1\} \to \mathbb{R}$ be the obvious injection. Then for all $a, \alpha \in \mathcal{P}(A)$, regarded as elements of $\mathcal{C}(A,I)$ and $\mathcal{C}(I,A)$, respectively, we have

$$p(a \circ \alpha) = \begin{cases} 1 & \text{if } a \cap \alpha \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Thus, $[\alpha] \in \mathbb{R}^{\mathcal{P}(A)}$ is the characteristic function of the set $[\alpha] = \{a \subseteq A | a \cap \alpha \neq \emptyset\}$. We can regard this as a kind of *possibility measure* on $\mathcal{P}(A)$, in the sense that $[\alpha](a) = 1$ iff $a$ is possible, given that $\alpha$ is certain. $V_o(A)$ is the span of these possibility measures in $\mathbb{R}^{\mathcal{P}(A)}$ — a space it would be nice to characterize more directly.

**Example 3.7 (Categories with Very Small Hom Sets)** Let $\mathcal{C}$ be any SMC such that $\mathcal{C}(A,B)$ is finite for all objects $A, B \in \mathcal{C}$ — for instance, any sub-category of the category of finite sets and relations. Let $S = \mathcal{C}(I,I)$, and let $R : S \to \mathbb{R}^S$ be the usual right action, given by $R_s(f)(x) = f(xs)$ for all $x, s \in S$ and all $f \in \mathbb{R}^S$. Since $R^S$ is finite-dimensional and $R_s$ is linear, we have a canonical monoid homomorphism $p : S \to \mathbb{R}_+$, namely $p(s) = |\det R_s|$, and hence, a canonical representation $V_o$.

# 4 Monoidality

We now adddress the questions of how $V_o$ interacts with the monoidal structure of $\mathcal{C}$. We begin with the observation that the construction $V_o(A), V_o(B) \mapsto V_o(A \otimes B)$ is well-defined: the assignment $A \mapsto \mathcal{C}(A,I)$ is injective, and we can read off $\mathcal{C}(A,I)$ as the domain of any function in $V(A) = \mathbb{R}^{\mathcal{C}(A,I)}$. Thus, given two spaces $V_o(A)$ and $V_o(B)$ in $V_o(\mathcal{C})$, we can unambigously define $V_o(A) \circledast V_o(B) := V_o(A \otimes B)$. Ultimately, we wish to invoke Lemma 2.1 to conclude that $V_o$ is the object part of a monoidal structure on $V_o(\mathcal{C})$, with respect to which $V_o$ a monoidal functor. This requires that the construction

$$V_o(\phi), V_o(\psi) \ \mapsto \ V_o(\phi \otimes \psi)$$

also be well-defined for all morphisms $\phi, \psi$ in $\mathcal{C}$. To insure this, additional constraints on $\mathcal{C}$ seem to be needed. One sufficient condition is that (i) the tensor unit $I$ be *separating*, meaning that the functor $\mathcal{C}(-,I)$ is injective on morphisms, and (ii) the monoid homomorphism $p : S \to \mathbb{R}$ is also injective. However, the injectivity of $p$ is a very strong constraint, not satisfied, for example, in the case of **FdHilb**. Thus, we need to say something more in order to secure the monoidality of $V_o$.

Before turning to this, however, we show that $V_o(A \otimes B)$ is indeed a product of $V_o(A)$ and $V_o(B)$ — or, more exactly, that the dual pair $(V_o(A \otimes B), V_o^{\#}(A \otimes B))$ is a product of the dual pairs $(V_o(A), V_o^{\#}(A))$ and $(V_o(B), V_o^{\#}(B))$ — in the sense of Definition 2.2. Recall [7] that in any symmetric monoidal category, we have, for every $\omega \in \mathcal{C}(I, A \otimes B)$, a natural mapping $\widetilde{\omega} : \mathcal{C}(B,I) \to \mathcal{C}(I,A)$, given by $\widetilde{\omega}(b) = (\mathrm{id}_A \otimes b) \circ \omega$ Dually, if $f \in \mathcal{C}(A \otimes B, I)$, there is a natural mapping $\widetilde{f} : \mathcal{C}(I,A) \to \mathcal{C}(B,I)$ given by $\widetilde{f}(a) = f \circ (a \otimes \mathrm{id}_B)$.

**Proposition 4.1** *For any objects $A$ and $B$ of $\mathcal{C}$, there exist canonical positive bilinear and linear mappings*

$$\circledast : V_o(A) \times V_o(B) \to V_o(A \otimes B) \quad \textit{and} \quad \Lambda : V_o(A \otimes B) \to \mathcal{B}(V_o^{\#}(A), V_o^{\#}(B))$$

*making $(V_o(A \otimes B), V_o^{\#}(A \otimes B))$ a product of ordered dual pairs in the sense of Definition 2.4.*

*Proof:* There is only one candidate for $\circledast$: it must be the bilinear extension — unique if it exists! — of the mapping

$$[\alpha], [\beta] \mapsto [\alpha] \circledast [\beta] := [\alpha \otimes \beta].$$

To see that this last is well-defined, let $\alpha \in \mathcal{C}(I,A)$, $\beta \in \mathcal{C}(I,B)$ and $f \in \mathcal{C}(A \otimes B, I)$: then

$$[\alpha \otimes \beta](f) = p(f \circ (\alpha \otimes \beta)) = p((f \circ (\mathrm{id}_A \otimes \beta)) \circ \alpha) = p(\widetilde{f}(\beta) \circ \alpha) = [\alpha](\widetilde{f}(\beta)). \tag{2}$$

This depends only on $[\alpha]$. In a similar way, one sees that $[\alpha \otimes \beta] = [\beta](\widetilde{f}(\alpha))$. Thus $[\alpha \otimes \beta]$ depends only on $[\alpha]$ for fixed $\beta$, and only on $[\beta]$ for fixed $\alpha$ — and hence, only on $[\alpha]$ and $[\beta]$.

Next, we must show that $[\alpha], [\beta] \mapsto [\alpha \otimes \beta]$ extends to a bilinear mapping $V_o(A) \times V_o(B) \to V_o(A \otimes B)$ (which will automatically be positive, if it exists). This amounts to showing that $\sum_i t_i[\alpha_i], \sum_j s_j[\beta_j] \mapsto \sum_{i,j} s_i t_j[\alpha_i \otimes \beta_j]$ is well-defined. Let $\rho = \sum_i t_i[\alpha_i \otimes \beta] \in V_o(A \otimes B)$. Then, for every $f \in \boldsymbol{\mathcal{C}}(A \otimes B, I)$, using (3), we have

$$\rho(f) = \sum_i t_i[\alpha_i \otimes \beta](f) = \sum_i t_i[\alpha_i](\widetilde{f}(\beta))$$

which depends only on $\sum_i t_i[\alpha_i] \in V_o(A)$. A similar argument shows that $\sum_i s_j[\alpha \otimes \beta_j]$ depends only on $\sum_j s_j[\beta_j]$ for fixed $\alpha$, whence, $\sum_{i,j} t_i s_j[\alpha_i \otimes \beta_j]$ depends only on $\sum_i t_i[\alpha_i]$ and $\sum_j s_j[\beta_j]$.

So much for $\circledast$. In order to define $\Lambda$, note that for every $\mu \in V_o(A \otimes B) \leq \mathbb{R}^{\boldsymbol{\mathcal{C}}(A \otimes B, I)}$ we have a natural linear mapping

$$\widetilde{\mu} : \boldsymbol{\mathcal{C}}(A, I) \to \mathbb{R}^{\boldsymbol{\mathcal{C}}(B,I)} \ \text{ given by } \ \widetilde{\mu}(a) : b \mapsto \mu(a \otimes b).$$

For $\mu = [\omega]$, $\omega \in \boldsymbol{\mathcal{C}}(I, A \otimes B)$, we have

$$\widetilde{[\omega]}(a) = [\widetilde{\omega}(a)] = [(a \otimes \mathrm{id}_B) \circ \omega] \in V_o(B). \tag{3}$$

As vectors of the form $\mu = [\omega]$ span $V_o(A \otimes B)$, we have a natural mapping $\widetilde{\mu} : \boldsymbol{\mathcal{C}}(A, I) \to V_o(B)$ for every $\mu \in V_o(A \otimes B)$. Dualizing, we have a linear mapping $\widetilde{\mu}^* : V_o(B)^* \to \mathbb{R}^{\boldsymbol{\mathcal{C}}(A,I)}$. Restricting this to $V_o^{\#}(B)$, and noting that for $\mu = [\omega]$, $\omega \in \boldsymbol{\mathcal{C}}(I, A \otimes B)$ and $b \in \boldsymbol{\mathcal{C}}(B, I)$,

$$\widetilde{\mu}^*(b) = [(\mathrm{id}_A \otimes b) \circ \omega] \in V_o(A), \tag{4}$$

we have $\widetilde{\mu}^* \in \mathcal{L}(V_o^{\#}(A), V_o(B))$. This gives us a natural positive linear mapping

$$\Lambda : V_o(A \otimes B) \to \mathcal{B}(V_o^{\#}(A), V_o^{\#}(B)) \ \text{ given by } \ \Lambda(\mu)(a, b) = \widetilde{\mu}^*(b)(a)$$

for all $\mu \in V_o(A \otimes B)$, $a \in V_o^{\#}(A)$ and $b \in V_o^{\#}(B)$.

It remains to verify conditions (ii)(a) and (ii)(b) of Definition 2.4. For the latter, notice that, by (2), $\Lambda(\mu)(-, b) \in V_o(A)_+$ for all $b \in V_o^{\#}(B)_+$. By (3), plus the fact that vectors of the form $a \in \boldsymbol{\mathcal{C}}(A, I)$ generate $V_o^{\#}(A)_+$, we also have $\Lambda(\mu)(a, -) \in V_o(B)_+$ for all $a \in V_o^{\#}(A)_+$. For the former, observe that, for all $\alpha \in \boldsymbol{\mathcal{C}}(I, A), a \in \boldsymbol{\mathcal{C}}(A, I)$, $\beta \in \boldsymbol{\mathcal{C}}(I, B)$ and $b \in (B, I)$, we have

$$\begin{aligned} \Lambda([\alpha] \circledast [\beta])(a, b) &= [\alpha \otimes \beta](a \otimes b) \\ &= p((\alpha \otimes \beta) \circ (a \otimes b)) \\ &= p((\alpha \circ a) \otimes (\beta \circ b)) \\ &= p(\alpha \circ a) p(\beta \circ b) = [\alpha](a)[\beta](b). \ \square \end{aligned}$$

We now return to the question of whether the product $V_o(A), V_o(B) \mapsto V_o(A) \circledast V_o(B)$ is the object part of a monoidal structure on $\boldsymbol{\mathcal{C}}$. At present, I can't show that this is always the case. We do, however, have two sufficient conditions. One of these is local tomography of $V_o(\boldsymbol{\mathcal{C}})$:

**Proposition 4.2** *If the product $(V_o(A \otimes B), V_o^{\#}(A \otimes B))$ is locally tomographic for all $A, B \in \boldsymbol{\mathcal{C}}$, then $V_o$ is monoidal, i.e., a representation in the sense of Definition 2.3.*

*Proof sketch:* Appealing to Lemma 2.1, we need only show that $V_o(\phi), V_o(\psi) \mapsto V_o(\phi \otimes \psi)$ is well-defined for $\phi \in \boldsymbol{\mathcal{C}}(A, B)$ and $\psi \in \boldsymbol{\mathcal{C}}(C, D)$ for all objects $A, B, C, D \in \boldsymbol{\mathcal{C}}$. Let $\phi \in \boldsymbol{\mathcal{C}}(A, B)$. We have a well-defined dual mapping $V_o(\phi)^{\#} : V_o^{\#}(B) \to V_o^{\#}(A)$, given by

$$(V\phi)^{\#}(e_b)(\alpha) = V(\phi)(\alpha)(b) = p(b \circ \phi \circ \alpha).$$

Now, if $\boldsymbol{\mathcal{C}}$ is locally tomographic, then $\Lambda : V(A \otimes B) \leq B(V_o^{\#}(A), V_o^{\#}(B))$, so that $V(\phi \otimes C)(\omega)$ is determined by values of

$$\begin{aligned} V_o(\phi \otimes C)(\omega)(a \otimes b) &= p((a \otimes b) \circ (\phi \otimes C) \circ \omega) \\ &= p(((a \circ \phi) \otimes b) \circ \omega) \\ &= V_o(\phi \otimes C)^{\#}(a \otimes b)(\omega) = (V_o(\phi)^{\#}(a) \otimes b)(\omega). \end{aligned}$$

As the right-hand side depends only on $V_o(\phi)^{\#}$, and hence, on $V_o(\phi)$, rather than on $\phi$, the mapping $V_o(\phi) \mapsto V_o(\phi \otimes C) : V_o(A \otimes C) \to V_o(B \otimes C)$ is well-defined. A similar argument shows that $V_o(A \otimes \psi)$ depends only on $V_o(\psi)$. Thus, we have a well-defined mapping $V_o(\phi), V_o(\psi) \mapsto V_o(\phi \otimes \psi)$. $\Box$

As noted above, all of our benchmark categories – **FdHilb**, **FRel**, etc. — are locally tomographic. However, one can have $V_o$ monoidal in the absence of local tomography, as in the case of real Hilbert space. On the other hand, all of our *finite-dimensional* examples, including **Rel**, are compact closed. Since all morphisms in such a category are represented by states, this is also sufficient:

**Proposition 4.3** *If $\mathcal{C}$ is compact closed, then $V_o$ is the object part of a monoidal functor.*

*Proof sketch:* Since $\mathcal{C}$ is compact closed, there is a natural mapping $\mathcal{C}(I, A^* \otimes B)$ to $\mathcal{C}(A, B)$ given by

$$\omega \mapsto \widehat{\omega} = (\epsilon_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \omega).$$

Moreover, every morphism in $\mathcal{C}(A, B)$ arises in this fashion. If $\omega \in \mathcal{C}(I, A^* \otimes B)$ and $\mu \in \mathcal{C}(I, C^* \otimes D)$, then one finds that $\widehat{\omega} \otimes \widehat{\mu} = \widehat{\omega \odot \mu}$ where $\omega \odot \mu := \tau \circ (\omega \otimes \mu)$ and and $\tau = \mathrm{id}_{A^*} \otimes \sigma_{B,C^*} \otimes \mathrm{id}_D$. Now if $\omega, \omega' : I \to A^* \otimes B$ and $\mu : I \to C^* \otimes D$, $[\omega] = [\omega'] \Rightarrow [\omega \otimes \mu] = [\omega' \otimes \mu]$ (see Equation (1) in the proof of Proposition 4.1). Letting $\phi = (\epsilon_{(A \otimes B)^*} \otimes \epsilon_{C \otimes D}) \circ \tau$, we have

$$\begin{aligned} V_o(\widehat{\omega} \otimes \widehat{\mu}) = V_o(\widehat{\omega \odot \mu}) &= V_o(\phi \circ (\omega \otimes \mu)) \\ &= V_o(\phi) \circ V_o(\omega \otimes \mu) \\ &= V_o(\phi) \circ V_o(\omega' \otimes \mu) \\ &= V_o(\phi \circ (\omega' \otimes \mu)) = V_o(\widehat{\omega' \odot \mu}) = V_o(\widehat{\omega'} \otimes \widehat{\mu}). \end{aligned}$$

A similar computation in the other argument shows that, for $\phi \in \mathcal{C}(A, B)$ and $\psi \in \mathcal{C}(C, D)$, $V_o(\phi \otimes \psi)$ depends only on $V_o(\phi)$ and $V_o(\psi)$. $\Box$

# 5 Normalization

To this point, we have made no attempt to distinguish between normalized and non-normalized states. From the convex-operational point of view, only normalized and sub-normalized states represent actual states of affairs; super-normalized states are a mathematical convenience. In order to make this distinction in the present context, we introduce some new structure.

**Definition 5.1** Let $V$ be an ordered representation of a symmetric monoidal category $\mathcal{C}$. An *unit* for $V$ is a choice, for each $A \in \mathcal{C}$, of a strictly positive functional $u_A \in V(A)^*$, such that

(i) For every $a \in \mathcal{C}(A, I)$, there exists some $t \in \mathbb{R}$, $t \geq 0$, such that $a \leq tu$;

(ii) for all $\alpha \in \mathcal{C}(I, A)$ and $\beta \in \mathcal{C}(I, B)$, $u_{A \otimes B}(\alpha \otimes \beta) = u_A(\alpha) \otimes u_B(\beta)$.

*Remarks:* (a) If $V(I) = \mathbb{R}$, we can interpret an unit as a natural transformation $u : V \to 1$, where 1 is the trivial representation $1(A) = \mathbb{R}$ for all objects $A$ and $1(\phi) = \mathrm{id}_{\mathbb{R}}$ for all $\phi \in \mathcal{C}(A, B)$. (b) $\mathcal{C}$ is a *discard category* [11] if every object $A \in \mathcal{C}$ is equipped with a morphism $\dot{\top}_A : A \to I$ such that $\dot{\top}_A \otimes \dot{\top}_B = \dot{\top}_{A \otimes B}$. In this case, $u_A := V(\dot{\top}_A)$ will supply a unit, *provided* this satisfies condition (i) above.

While the existence of a unit is not guaranteed, the usual examples have *canonical* units: In **FRel**, where $\mathcal{C}(A, *) \simeq \mathcal{P}(A)$, there is a natural unit, namely $u_A = A$ — or rather, $u_A([\alpha]) = [\alpha](A) = p(\alpha \cap A) = 1$ for all $\alpha \in \mathcal{C}(*, A)$. In **FdHilb**, where $V(A)$ is the space of hermitian operators on $A$, the trace is a unit. If $u$ is a unit for a representation $V$ of $\mathcal{C}$, Condition (i) guarantees that $u_A$ is strictly positive on $V(A)_+$, meaning that $u_A(\alpha) > 0$ for all nonzero $\alpha \in V(A)_+$, for all $A \in \mathcal{C}$. It follows that the set $\Omega(A) := u^{-1}(1) \cap V(A)_+$ is a *base* for the cone $V(A)_+$, i.e., every $\alpha \in V(A)_+$ is uniquely a non-negative multiple of a point in $\Omega(A)$ ([5], Theorem 1.47) If $V(A)$ is finite-dimensional, this guarantees that $u$ is an *order unit* for $V(A)^*$ (as this is spanned by functionals $a \in \mathcal{C}(A, I)$), and we can regard $(V(A), V(A)^*, u_A)$

as a COM. In the general case, however, the matter is more delicate. Condition (ii) in Definition ... does guarantee that $u_A$ will be an order unit for $V_o(A)^{\#}$, *if* $u_A$ belongs to this space, i.e., is a linear combination of functionals corresponding to elements of $\mathcal{C}(A, I)$. It's worth recording the following corollary to Thm...

**Corollary 5.2** *Let $u$ be a unit for $V_o$, such that $u_A \in V_o^{\#}(A)$ for all $A \in \mathcal{C}$. Then*

- *(a) $(V_o(A), V_o^{\#}(A), u_A)$ is a COM for every $A \in \mathcal{C}$, and*
- *(b) $(V_o(A \otimes B), V_o^{\#}(A \otimes B), u_{A \otimes B})$ is a non-signaling composite of $(V_o(A), V_o^{\#}(A), u_A)$ and $(V_o(B), V_o^{\#}(B), u_B)$ for every $A, B \in \mathcal{C}$.*

As we will see below, however (cf. Example 5.9), $u_A$ need not belong to $V_o^{\#}(A)$. In this situation, one can enlarge both $V_o(A)$ and $V_o(A)^{\#}$ so as to obtain a COM $(V(A), V^{\#}(A), u_A)$. Given a unit $u$ for $V_o$, we define an *effect* to be an element $a \in V_o(A)^*$ with $0 \leq a \leq u_A$ (in the dual ordering). As discussed earlier, an effect represents a mathematically possible measurement-outcome, since, for any normalized state $0 \leq a(\alpha) \leq 1$, so that we can regar $a(\alpha)$ as a probability. We write $[0, u_A]$ for the set of effects of $A$. There is a natural linear mapping $\mathbf{ev} : V_o(A) \to \mathbb{R}^{[0,u_A]}$, given by evaluation (that is, $\mathbf{ev}(\rho)(a) = a(\rho)$ for $a \in [0, u_A]$ and $\rho \in V_o(A)$). By condition (i) in the definition of a unit, this mapping is injective. Henceforward, we shall identify $V_o(A)$ with its image under $\mathbf{ev}$, that is, we regard $V_o(A)$ as a subspace of $\mathbb{R}^{[0,u_A]}$. Let $\Omega(A, u)$ denote the closure of $\Omega_o(A, u)$ in the product topology on $\mathbb{R}^{[0,u_A]}$, noting that this is a *compact* convex set.

**Definition 5.3** *Let $V(A)$ be the span of $\Omega(A, u)$ in $\mathbb{R}^{[0,u_A]}$, ordered by the cone $V(A)_+$ consisting of nonnegative multiples of points of $\Omega(A, u)$. That is, $V(A)_+ := \{t\alpha \mid \alpha \in \Omega(A, u) \text{ and } t \geq 0\}$.*

Since $\Omega(A, u)$ is compact, it follows that $V(A)$ is complete in the *base norm*, i.e, the Minkowski functional of the convex hull of $\Omega(A, u) \cup -\Omega(A, u)$. For details, see [4].

**Lemma 5.4** *The positive cone $V(A)_+$ of $V(A)$ is the pointwise closure, in $\mathbb{R}^{[0,u_A]}$, of $V_o(A)_+$.*

*Proof:* Let $K$ denote the closure of $V_o(A, u_A)$ in $\mathbb{R}^{[0,u_A]}$. Clearly, $V(A)_+ \subseteq K$. To establish the reverse inclusion, choose a net $t_i\rho_i \in V_+(A, u)$ with $\rho_i \in \Omega(A, u)$, and suppose $t_i\rho_i \to \rho \in \mathbb{R}^{[0,u_A]}$. Note that $\rho(a) \lim_i t_i\rho_i(a) \geq 0$ for all $a \in [0, u_A]$. Note, too, that if $\rho(a) > 0$ for some $a$, then eventually we must have $t_i\rho_i(a) > 0$, whence, $\rho_i(a) > 0$. Now, $t_i = u_A(t_i\rho_i) \to u_A(\rho) = \rho(u_A)$. If $u_A(\rho) = 0$, then $t_i \to 0$. I claim this implies $\rho = 0$. If not, then for some $a \in [0, u_A]$ and $0 < \epsilon < \rho(a)$, we eventually have $t_i\rho_i(a) > \rho(a) - \epsilon$, whence, $t_i = (t_i\rho_i)(u_A) \geq t_i\rho_i(a) > \rho(a) - \epsilon$, contradicting the fact that $t_i \to 0$. Now suppose $\rho(u_A) > 0$. Then we may write $\rho$ as $t\nu$ where $\nu = \rho/u_A(\rho)$ and $t = u_A(\rho)$; we then have $t_i = t_i\rho_i(u_A) \to t = \rho(u_A)$ and $t_i\rho_i(a) \to t\rho(a)$, whence, $\rho_i(a) \to \rho(a)$, for every $a \in [0, u_A]$. Thus, $\rho \in V_+(A)$. □

**Definition 5.5** *Let $V^{\#}(A, u)$ denote the span in $V(A)^*$ of the evaluation functionals associated with points $a \in [0, u_A]$.*

*Notation:* From this point forward, let's assume a fixed unit $u$ is given, and, accordingly, abbreviate $V(A, u)$ as $V(A)$ and $V^{\#}(A, u)$ as $V^{\#}(A)$. Also, wherever it seems safe to do so, let's write $a(\alpha)$ for $\alpha(a)$, conflating $a \in [0, u_A]$ with the corresponding evaluation functional in $V(A)^*$.

**Lemma 5.6** $(V(A), V^{\#}(A), u_A)$ *is a convex operational model.*

*Proof:* $V^{\#}(A)$ is a separating space of functionals on $V(A)$, and, by construction, $u_A$ is an order unit in $V^{\#}(A)$. □

In any ordered abelian group, an interval $[0, u]$ is an *effect algebra* [14] under the partial operation $a \oplus b = a + b$ (defined for $a, b \in [0, u]$ provided that $a + b$ is again in $[0, u]$). In particular, for every $A \in \mathcal{C}$, $[0, u_A] \subseteq V_o(A)^*$ is an effect algebra.

**Definition 5.7** *A (finitely additive)* measure *on an effect algebra $L$ is a mapping $\mu : L \to \mathbb{R}_+$ such that, for all $a, b \in L$, $a \perp b \;\Rightarrow\; \mu(a \oplus b) = \mu(a) + \mu(b)$ (where $a \perp b$ means that $a \oplus b$ is defined). A* signed measure *on $L$ is a difference of measures.*

The set $M(L)$ of all signed measures on $L$ is a complete base-normed space; the dual order-unit is given by $u(\mu) = \mu(1_L)$ where $1_L$ is the unit element of $L$. It is easy to see that every element of $V(A, u)$ is a measure on the effect algebra $[0, u_A]$. Hence, $V(A)$ is a closed subspace of $M(A) := M([0, u_A])$.

**Lemma 5.8** $(V^{\#}(A))^{\star} = M(A)$.

The proof is essentially identical to that of the generalized Gleason theorem of Busch [8]; see also [9].

**Example 5.9** Let $\mathcal{C} = \mathbf{Hilb}$, so that each $A \in \mathcal{C}$ is a complex Hilbert space. As discussed in Example 3.5, $V_o(A)$ is the space of finite-rank Hermitian operators on $A$. Let $u_A \in V_o(A)^*_+$ be the trace: this is an unit for $V_o$ in the sense of Definition 5.1 . Any positive linear functional $a \in V_o(A)^*$ with $a(\rho) \leq \mathrm{Tr}(\rho)$ for all $\rho \in V_o(A)$ is bounded with respect to the trace norm on $V_o(A)$. Since $V_o(A)$ is trace-norm dense in the space $V_1(A)$ of trace-class Hermitian operators, $a$ extends uniquely to a bounded linear functional on $V_1(A)$, whence, corresponds to a bounded Hermitian operator on $A$. Thus, $[0, u_A]$ is the standard interval of effects for $A$, and $V^{\#}(A)$ is the space of bounded self-addjoint operators. Finally, the extension of Gleason's Theorem to finitely additive measures on $[0, u_A]$ allows us to identify $M(A)$ with the space spanned by the the set of finitely additive states on the factor $\mathcal{B}(A)$. Since the state space of $\mathcal{B}(A)$ is the weak-$*$ closed convex hull of the set of vector states, i.e., of the set of pure states in $V_o(A)$ (see, e.g., [17] Corollary 4.3.10), we have $V(A) = M(A)$.

This suggests the following

**Definition 5.10** *Let $\Omega_1(A, u)$ denote the closure of $\Omega_o(A, u)$ in the base norm on $V(A, u)$, and let $V_1(A, u)$ be the closed subspace of $V(A)$ spanned by $\Omega_1$ (and ordered by $V_1(A)_+ = V_1(A) \cap V_+(A)$).*

It is easy to see that $M(A)$ is pointwise-closed in $\mathbb{R}^{[0,u_A]}$, whence, we have natural embeddings

$$V_o(A) \leq V_1(A) \leq V(A) \leq (V^{\#}(A))^{\star} = M(A) \leq \mathbb{R}^{[0,u_A]}.$$

Since the choice of $u_A$ is not canonical, we can't expect $V, M$ or $V_1$ to be functorial on $\mathcal{C}$. However, we can single out the sub-category of $\mathcal{C}$ having the same objects, but only those morphisms respecting $u_A$:

**Definition 5.11** *$\mathcal{C}_u(A, B)$ consists of those morphisms $\phi \in \mathcal{C}(A, B)$ such that $(V_o\phi)^*([0, u_B]) \subseteq [0, u_A]$ — equivalently, such that $V_o(\phi)^*(u_B) = u_B \circ V_o(\phi) \leq u_A$.*

Clearly, the composite of $f \in \mathcal{C}_u(A, B)$ and $g \in \mathcal{C}_u(B, C)$ yields a morphism in $\mathcal{C}_u(A, C)$, so we have here a sub-category, $\mathcal{C}_u$, of $\mathcal{C}$. Moreover, since $\phi \in \mathcal{C}_u(A, B)$ implies that $V_o(\phi)^*([0, u_B]) \subseteq [0, u_A]$, we have a functor $M : \mathcal{C}_u \to \mathbf{OrdLin}$ given by

$$M(\phi)(\mu) = \mu \circ V_o(\phi)^*$$

where $\mu \in M(A)$ and $\phi \in \mathcal{C}(A, B)$. In fact, $V$ and $V_1$ are also functorial with respect to $\mathcal{C}_u$:

**Lemma 5.12** *$A \mapsto V(A)$ and $A \mapsto V_1(A)$ are the object parts of functors $V, V_1 : \mathcal{C}_u \to$ **OrdLin**,*

*Proof sketch:* Let $\phi \in \mathcal{C}_u(A, B)$. Then if $b \in [0, u_B]$, we have

$$V_o(\phi)^*(b)(\rho) = b(V_o(\phi)(\rho)) \leq u_B(V_o(\phi)(\rho)) \leq u_A(\rho)$$

for all $\rho \in V_o(A)_+$. Thus, $V_o(\phi)^*(b) \in [0, u_A]$. We now have a continuous mapping $\mathbb{R}^{[0,u_A]} \to \mathbb{R}^{[0,u_B]}$, namely $V_o(\phi)^{**} : \rho \mapsto \rho \circ V(\phi)^*$. It is straightforward that this mapping takes $\Omega_o(A, u_A)$ into $V_o(B, u_B)_+$; as it preserves effect-wise limits, we it takes the effect-wise closure, $\Omega(A, u_A)$, of $\Omega_o(A, u_A)$ into the effect-wise closure of $V_o(A, u_A)_+$, which, by Lemma 5.10, is $V(A, u)_+$. This gives us the desired positive linear mapping $V(\phi) : V(A) \to V(B)$. To define $V_1(\phi)$, observe that since $u_B(V(\phi)(\alpha)) \leq u_A(\alpha)$ for all $\alpha \in \Omega(A, u)$, we have $\|V(\phi)\| \leq 1$, where $\| \cdot \|$ denotes the operator norm, computed relative to the base norms on $V(A)$ and $V(B)$. In particular, $V(\phi)$ is bounded, hence, continuous, with respect to these norms. Since $V(\phi)$ takes $V_o(A)_+$ into $V_o(B)_+$, it takes the span of the base-norm closure of the former cone to that of the latter, i.e, maps $V_1(A)$ into $V_1(B)$, giving us the desired positive linear mapping $V_1(\phi)$. □

**Corollary 5.13** *If $V_o$ is monoidal, so are $V$ and $V_1$. In particular, if $V_o(\boldsymbol{\mathcal{C}})$ is locally tomographic or $\boldsymbol{\mathcal{C}}$ is compact closed, then both $V$ and $V_1$ are monoidal functors.*

*Proof:* Let $\phi \in \boldsymbol{\mathcal{C}}(A, C)$ and $\psi \in \boldsymbol{\mathcal{C}}(B, D)$. We wish to show that $V(\phi \otimes \psi)$ depends only on $V(\phi)$ and $V(\psi)$. Let $\rho \in V_o(A \otimes B)$, $f \in [0, u_{CD}]$. We have $V_o$ is monoidal, then $V_o(\phi \otimes \psi)$ depends only upon $V_o(\phi)$ and $V_o(\psi)$. $V(\phi \otimes \psi)(\rho)(b) = \rho(b \circ V_o(\phi \otimes \psi))$ . But, if $V_o$ is monoidal, $V_o(\phi \otimes \psi)$ depends only on $V_o(\phi)$ and $V_o(\psi)$ — whence, only on $V(\phi)$ and $V(\psi)$ (since $V(\phi) = V(\phi')$ implies $V_o(\phi) = V_o(\phi')$). The case of $V_1$ follows. □

It remains to ask whether the pairs $(V(A \otimes B), V^{\#}(A \otimes B), u_{A \otimes B})$ and $(V_1(A \otimes B), V_1^{\#}(A \otimes B), u_{A \otimes B})$ are respectable non-signaling composites, in the sense of definition 2.4. I believe this to be the case, but do not have a proof. For the time being, I leave this as a conjecture.

# 6 Further Questions

This has been only a preliminary excursion into what looks like a rather large territory, raising many more questions than have been settled. Besides the conjecture mentioned above, a very partial list of unfinished business includes: (1) How are representations $V_o$ arising from various different monoid homomorphisms $\boldsymbol{\mathcal{C}}(I, I) \to \mathbb{R}_+$ related to one another? (2) If $\boldsymbol{\mathcal{C}}$ is dagger compact, and $V : \boldsymbol{\mathcal{C}} \to \mathbf{RVec}$ is a monoidal representation, will the category $V(\boldsymbol{\mathcal{C}})$ be weakly self-dual in the sense of [7]? (3) How do the constructions sketched above (notably, $V_o$) interact with Selinger's CPM construction [20]? (4) Can one characterize abstractly those symmetric monoidal categories $\boldsymbol{\mathcal{C}}$ for which (there exists a monoid homomorphism $p : \boldsymbol{\mathcal{C}}(I, I) \to \mathbb{R}_+$ such that) $\boldsymbol{\mathcal{C}}$ is isomorphic, or equivalent, to $V_o(\boldsymbol{\mathcal{C}})$? (5) What is the connection between the constructions described here and the approach to constructing operational models based on Chu spaces, explored in [1, 3]?

# References


[1] S. Abramsky, Big toy models: representing physical systems as Chu spaces, `arXiv:0910.2393` (2009).

[2] S. Abramsky and B. Coecke, Categorical quantum mechanics, in *Handbook of Quantum Logic and Quantum Structures* **II**, K. Engesser, D. Gabbay, D. Lehman, eds., Elsevier, 2008.

[3] S. Abramsky and C. Heunen, Operational theories and categorical quantum mechanics, in J. Chubb and A. Eskandarian, *Logic and algebraic structures in quantum computing and information, Cambridge University Press*, Lecture Notes in Logic **45** (2016) 88-122 (`arXiv:1206.0921`)

[4] E. M. Alfsen and F. W. Shultz, *State Spaces of Operator Algebras: Basic Theory, Orientations, and $C^*$-products.* Birkhauser. 2002.

[5] C. D. Aliprantis andd R. Tourky, *Cones and Duality*, Graduate Texts in Mathematics vol. **84**, American Mathematical Society, 2007.

[6] H. Barnum and A. Wilce, Post-classical probability theory, in G. Chiribella and R. Spekkens, Eds., *Quantum Theory: Informational Foundations and Fois*, Springer, 2016; `arXiv:1205.3833`

[7] H. Barnum, R. Duncan and A. Wilce, Symmetry, compact closure, and dagger-compactness for categories of convex operational models, J. Phil. Logic **42** (2013), 501-523; `arXiv:1004.2920`

[8] P. Busch, Quantum states and generalized observables: a simple proof of Gleason's Theorem, Phys. Rev. Lett. **91** (2003); `arXiv: 9909073`

[9] C. Caves, C. Fuchs, K. K. Manne and J. Renes, Gleason-type derivations of the quantum probability rule for generalized measurements, Foundations of Physics **32** (2004) 193-209.

[10] K. Cho, B. Jakobs, B. Westerbaan and A. Westerbaan, An Introduction to Effectus Theory, `arXiv:1512.05813` (2015)

[11] B. Coecke and J. Selby, Leaks: quantum, classical, intermediate, and more, `arXiv:1701.07404`

[12] E. B. Davies and J. T. Lewis, An operational approach to quantum probability, Comm. Math. Phys. **17** 239-260 (1970).

[13] C. M. Edwards, The operational approach to algebraic quantum theory I, Comm. Math. Phys. **16** (1970) 207-230.

[14] D. J. Foulis and M. K. Bennett, Effect algebras and unsharp quantum logics, Foundations of Physics **24** (1994), 1331-1352.

[15] S. Gogioso and C. M. Scandolo, Categorical probabilistic theories, `1701.08075` (2017)

[16] P. Janotta and R. Lal, Generalized probabilistic theories without the no-restriction hypothesis, Phys. Rev. A 87, 052131 (2013); `arXiv:1302.2632`

[17] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, vol. I*, Academic Press, 1983.

[18] G. Ludwig, Z. Versuch einer axiomatischen Grundlegung der Quantenmechanik und allgemeinerer physikalischer Theorien, *Z. Physik* **181** (1964); Attempt of an Axiomatic Foundation of Quantum Mechanics and More General Theories II, *Comm. Math. Phys.* **4** (1967) , 331-348; Attempt of an Axiomatic Foundation of Quantum Mechanics and More General Theories, III Comm. Math. Phys. **9** (1968) 1-12.

[19] G. Mackey, Mathematical Foundations of Quantum Mechanics, Addison-Wesley, 1963

[20] P. Selinger, Dagger compact closed categories and completely positive maps (extended abstract), in Proceedings of QPL 05. Electronic Notes in Theoretical Computer Science **170**, 139-163 (2007).

[21] S. Tull, Operational theories of physics as categories, `arxiv:1602.06284`

[22] A. Wilce, A royal road to quantum mechanics (or thereabouts), `arXiv:1606.09306` (2016)