A categorical semantics for causal structure

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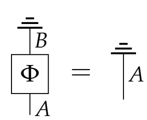
Original version accepted for LICS 2017. For a full version with proofs, see [9].

We present a categorical construction for modelling both definite and indefinite causal structures within a general class of process theories that include classical probability theory and quantum theory. Unlike prior constructions within categorical quantum mechanics, the objects of this theory encode fine-grained causal relationships between subsystems and give a new method for expressing and deriving consequences for a broad class of causal structures. To illustrate this point, we show that this framework admits processes with definite causal structures, namely one-way signalling processes, non-signalling processes, and quantum n-combs, as well as processes with indefinite causal structure, such as the quantum switch and the process matrices of Oreshkov, Costa, and Brukner. We furthermore give derivations of their operational behaviour using simple, diagrammatic axioms.

Symmetric monoidal categories (SMCs), especially compact closed ones, provide a framework for studying theories of interacting (physical) processes, a.k.a. *process theories* [5], such as quantum mechanics and classical probability theory. By adding a bit more structure, namely a *discarding* effect ⏚, one can study causal relationships between the inputs and outputs of a black-box process in such a theory. For instance, one can require that certain inputs/outputs be causally ordered before others, or that a process be non-signalling. The simplest such requirement is that a process be *causal* [3, 6], which intuitively means 'if we discard the output of a causal process, it is as if the process never happened in the first place':



(1)

This captures in a generic way the property of a matrix of positive numbers being stochastic (in the classical case) or of a completely positive map being trace preserving (in the quantum case). For a process to be non-signalling, it should additionally satisfy the following two equations: There exist causal Φ', Φ'' such that

(2)

which are the graphical version of the usual conditional independences expected for a non-signalling process. Indeed for stochastic matrices, these equations correspond precisely to the requirements $P(A'|AB) = P(A'|A)$ and $P(B'|AB) = P(B'|B)$ on a conditional probability distribution $P(A'B'|AB)$.

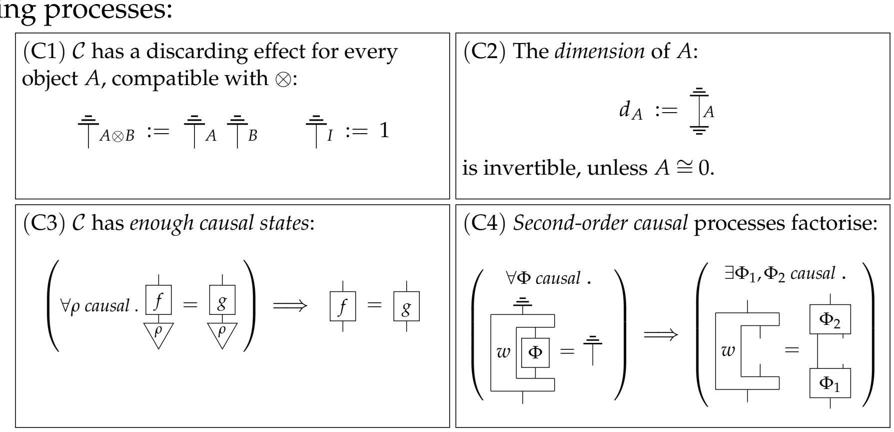
To appear in EPTCS.

There are many other causal structures one might wish to impose, e.g. causally ordered chains of input/output pairs, or arbitrary directed acyclic graphs. In addition to such processes, one can study *higher-order causal processes*, i.e. mappings from processes to processes that preserve certain causal structures. These give a formal basis to the study of multi-party protocols or games [7, 2] and more recently to the study of *indefinite causal structures* [10, 1, 4], where the act of causally-ordering a set of local processes (i.e. 'local laboratories') can itself be regarded as one of many higher-order processes which can be superposed.

However, compact closed categories are too crude for the study of such properly higher-order structures. Indeed the *distributivity isomorphism* $(A \otimes B)^* \cong A^* \otimes B^*$, present in a compact-closed category ensures that all second-order maps are also first-order maps, hence all higher-order structure collapses. If we remove this isomorphism, we obtain the weaker notion of a $*$-*autonomous category*, which exhibits precisely the rich higher-order structure needed for the study of causality. Hence, we take a suitable compact-closed base category $\mathcal{C}$ of all processes (causal and otherwise), and create a new $*$-autonomous category Caus$[\mathcal{C}]$—by a method similar to the so-called *double-gluing construction* [8]—of first- and higher-order causal processes. A particularly striking example of the relevance of the $*$-autonomous structure is the following. Due to the failure of the distributivity isomorphism, $*$-autonomous categories come naturally with two tensor-products: the usual tensor $\boldsymbol{A} \otimes \boldsymbol{B}$ and its De Morgan dual, or *par*, given by $\boldsymbol{A} \parr \boldsymbol{B} := (\boldsymbol{A}^* \otimes \boldsymbol{B}^*)^*$. While these coincide for first-order systems (i.e. state spaces), they give two distinct, and perfectly natural, ways in Caus$[\mathcal{C}]$ to combine the type $\boldsymbol{A} \multimap \boldsymbol{A}'$ of processes from $\boldsymbol{A}$ to $\boldsymbol{A}'$ with the type $\boldsymbol{B} \multimap \boldsymbol{B}'$ of processes from $\boldsymbol{B}$ to $\boldsymbol{B}'$:

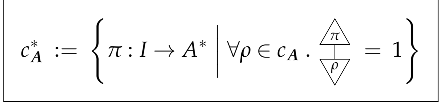
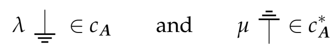
$$\textit{non-signalling processes} := (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \qquad \textit{all processes} := (\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$$

The Precausal Category $\mathcal{C}$. We begin by defining a suitable category of 'raw materials' from which to build a category of higher-order causal processes. A *precausal category* is a compact closed category, which satisfies four additional axioms concerning the behaviour of discarding processes:



The two main examples of precausal categories are: **CPM**, the category whose objects are spaces $\mathcal{B}(H)$ of bounded operators on finite dimensional Hilbert spaces and whose morphisms $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ are completely positive maps; and $\mathbf{Mat}(\mathbb{R}_+)$, the category whose objects are natural numbers and whose morphisms $f : m \to n$ are $n \times m$ matrices of positive numbers. The causal processes in these two categories are quantum channels and stochastic matrices, respectively.

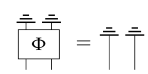
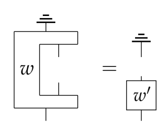
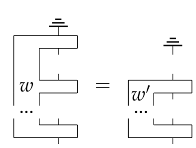
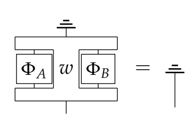
The *-Autonomous Category Caus$[\mathcal{C}]$. Our construction takes a precausal category $\mathcal{C}$ and creates a new category Caus$[\mathcal{C}]$ of first- and higher-order causal processes. It has as objects pairs $\boldsymbol{A} := (A, c_{\boldsymbol{A}})$ where A is an object of $\mathcal{C}$ and $c_{\boldsymbol{A}} \subseteq \mathcal{C}(I, A)$ is a set of states on A satisfying two additional requirements:

Closure: $c_{\boldsymbol{A}} = c_{\boldsymbol{A}}^{**}$, where:	*Flatness:* $\exists$ invertible $\lambda, \mu : I \to I$ such that
	

A morphism $f : \boldsymbol{A} \to \boldsymbol{B}$ in Caus$[\mathcal{C}]$ is a map $f : A \to B$ in $\mathcal{C}$ such that $\rho \in c_{\boldsymbol{A}} \Rightarrow f \circ \rho \in c_{\boldsymbol{B}}$.

The resulting category is $*$-*autonomous*. That is, Caus$[\mathcal{C}]$ is an SMC with a full and faithful functor sending each object $\boldsymbol{A}$ to its dual $\boldsymbol{A}^* := (A^*, c_{\boldsymbol{A}}^*)$ such that, for $\boldsymbol{A} \multimap \boldsymbol{B} := (\boldsymbol{A} \otimes \boldsymbol{B}^*)^*$, we have a natural isomorphism: Caus$[\mathcal{C}](\boldsymbol{A} \otimes \boldsymbol{B}, \boldsymbol{C}) \cong$ Caus$[\mathcal{C}](\boldsymbol{A}, \boldsymbol{B} \multimap \boldsymbol{C})$. The $*$-autonomous structure, namely $\otimes$, $\multimap$, and $\parr$, lets us define types for processes exhibiting many different causal structures.

Classification of (higher-order) causal processes. We can always canonically equip an object A in $\mathcal{C}$ with the set $c_{\boldsymbol{A}} := \{\bar{\top}_A\}^*$ of all causal states. We call systems of the form $\boldsymbol{A} := (A, \{\bar{\top}_A\}^*)$ *first-order systems*. Then, we can show that the full subcategory of first-order systems in Caus$[\mathcal{C}]$ is isomorphic to the subcategory of causal processes (i.e. those satisfy equation (1)) in $\mathcal{C}$. Using first-order systems as building blocks, we then build several more interesting higher-order systems, and prove that having the following types in Caus$[\mathcal{C}]$ is equivalent to each of the following operational characterisations in terms of $\bar{\top}$:

	Type	Operational Characterisation
causal processes	$\boldsymbol{A} \multimap \boldsymbol{A}'$	*equation* (1) *above*
non-signalling joint processes	$(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$	*equation* (2) *above*
all joint processes	$(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$	
one-way signalling processes *(a.k.a. quantum* 2*-comb)*	$\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}'$	
one-way signalling processes *(a.k.a. quantum n-comb)*	$\boldsymbol{A}_1 \multimap (\boldsymbol{A}_1' \multimap (\cdots) \multimap \boldsymbol{A}_n) \multimap \boldsymbol{A}_n'$	
Bipartite second order cuasal processes (SOC_2) *(e.g. OCB process matrix, quantum switch)*	$(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$	

We furthermore present n-fold versions of the characterisations above, namely n-party non-signalling and SOC_n.

We conclude by noting that, since the internal logic of $*$-autonomous categories is multiplicative linear logic, theorems about causal structures can be proved automatically using existing automated theorem provers such as `llprover` [11]. Indeed several of the results in this paper were discovered with the help of automated tools.

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