Classification of all alternatives to the Born rule in terms of informational properties

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1 Introduction

The extent to which the structure of measurements and probabilities is already encoded in the structure and dynamics of pure states of quantum theory has been the subject of much debate. There have been attempts to derive the Born rule (which assigns probabilities to measurement outcomes), however these are often deemed controversial [3, 6, 7]. In this work we suggest a more neutral approach where we consider all possible alternatives to the Born rule and explore the consequences of this change. We consider theories with the same pure states and dynamics as quantum theory but different measurement rules. We classify all these alternative theories using representation theoretic tools and describe informational properties of these alternatives. We show that no restriction of effects and bit symmetry single out the Born rule. We also conjecture that the Born rule is the only probabilistic assignment consistent with local tomography.

2 Alternative Born rules

The axioms of quantum mechanics postulate that pure states of a finite dimensional quantum system are rays on a complex vector space $\mathrm{P}\mathbb{C}^d$. The dynamical evolution of a closed system is given by $\psi \to U\psi$ with $U \in \mathrm{PU}(d)$ (the projective unitary group) and $\psi \in \mathrm{P}\mathbb{C}^d$. The outcomes of a measurement are associated to POVM elements $\{E_1, \dots, E_n\}$ which sum to the identity. The probability of obtaining an outcome E_i for a state ψ is given by the Born rule:

$$P(E_i|\psi) = \langle\psi|E_i|\psi\rangle \tag{1}$$

In this work we consider theories which share the dynamical structure of quantum theory, which is to say that pure states are given by $\mathrm{P}\mathbb{C}^d$ and dynamics by $\psi \to U\psi$ with $U \in \mathrm{PU}(d)$. However we allow for a different probabilistic structure where outcomes are no longer associated to POVM elements and where probabilities of outcomes are not determined by the Born rule. Rather we allow outcome probabilities to correspond to arbitrary functions $F : \mathrm{P}\mathbb{C}^d \to [0,1]$. A set of these functions $\{F_i\}$ form a measurement if $\sum_i F_i(\psi) = 1 \; \forall \psi \in \mathrm{P}\mathbb{C}^d$. We emphasize that no requirements are placed on these functions, they are not restricted to being powers of $|\langle E|\psi\rangle|$ (for an outcome associated to a projector $|E\rangle\langle E|$) as has been the

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case in previous work dealing with modified Born rules [1]. An alternative probabilistic structure is a set of these functions subject to some operational requirements, such as the composition of a measurement and a transformation also being a measurement.

We show that every alternative probabilistic structure (given by a set of outcome probability functions) is in correspondence with a representation of the unitary group. This is the representation of the dynamical group which acts linearly on the convex set of mixed states (the state space). When the probabilistic structure is given by the Born rule this is the adjoint representation. For the case of the qubit the state space is the Bloch ball acted on by the three-dimensional (adjoint) representation of $\mathrm{SU}(2)$. The state space of a system with pure states $\mathrm{P}\mathbb{C}^2$ and an alternative Born rule will be the convex hull of a 2-sphere embedded in a space of dimension $d \neq 3$ (the case $d = 3$ corresponding to the Born rule).

Moreover it is not the case that every representation of the unitary group is in correspondence with an alternative probabilistic structure. We find the representations of $\mathrm{PU}(d)$ which are in correspondence with a probabilistic structure and show the correspondence to be one-to-one.

3 Phenomenology of these alternative theories

We then make use of this classification to study these alternative theories, which have the same manifold of pure states and transformation group as quantum theory but different mixed states and measurements. We study informational properties of these alternative theories such as no-simultaneous encoding and the existence of phase groups for maximal measurements. We find that the property of pure state bit symmetry singles out the Born rule from all other alternatives. Pure state bit symmetry states that pairs of distinguishable pure states are related by a transformation belonging to the dynamical group [5]. In other words all logical bits are equivalent.

A central feature of any theory of nature is that of composition of systems. We find representation theoretic criteria for how systems with alternative Born rules compose in terms of branching rules. A branching rule describes how the representation of a group restricted to subgroup decomposes into irreducible representations of that subgroup.

The notion of local tomography can naturally be described in terms of branching rules. A composite system obeys local tomography if it can be fully characterised using only local measurements on its subsystems and correlations between local measurements [2, 4]. Quantum theory is locally tomographic. We determine the branching rules for a number of systems with alternative Born rules and observe that they do not have the decomposition compatible with local tomography. We make the conjecture that the only probabilistic structure compatible with the dynamical structure of quantum theory and local tomography is the Born rule.

4 Future work

In this work we classify a large family of theories which share the dynamical structure of quantum theory and find informational properties which distinguish them from quantum theory. This allows us to single out the Born rule. The methods used can be extended to describe arbitrary transitive probabilistic theories, which can be similarly studied and classified using representation theory. We give the example of theories with pure states given by a complex Grassmann manifold (a natural generalisation of projective space) and classify all possible probabilistic structures for these theories. We discuss how to use these representation theoretic methods to study arbitrary transitive probabilistic theories.

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