No-go theorems for functorial localic spectra of noncommutative rings

Benno van den Berg, Universiteit Utrecht
and Chris Heunen, University of Oxford

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Notion of spectrum: topological invariant of an algebraic structure. Often induces a *duality*.

Stone spectrum:

\[
\text{Bool}^{\text{op}} \leftrightarrow \text{Stone}
\]

Gelfand spectrum:

\[
\text{cCStar}^{\text{op}} \leftrightarrow \text{KHaus}
\]

Zariski spectrum:

\[
\text{cRing}^{\text{op}} \leftrightarrow \text{AffSch}
\]
Manuel Reyes ⇒ works in noncommutative algebra and algebraic geometry and is interested in extension of the Zariski spectrum for noncommutative rings.

\[
\begin{align*}
\text{cRing}^{\text{op}} & \xrightarrow{} \text{AffSch} \\
\text{Ring}^{\text{op}} & \xrightarrow{} ?? 
\end{align*}
\]

Noticed: different proposals of spectrum for noncommutative rings exist, but none is functorial. Wondered: Is there a reason for that?
Work of Reyes, continued

Of course, there is a similar problem for C*-algebras:

\[
\begin{align*}
\text{cCStar}^{\text{op}} & \rightarrow \text{KHaus} \\
\downarrow & \downarrow \\
\text{CStar}^{\text{op}} & \rightarrow ??
\end{align*}
\]

Then Reyes heard about the Kochen-Specker Theorem and Bohrification and things “clicked”.

### Theorem

Any functor $F : \text{Ring}^{\text{op}} \to \textbf{Top}$ (or $\textbf{Sets}$) which assigns the Zariski spectrum to a commutative ring must yield the empty space (or set) on $\mathbb{M}_n(\mathbb{C})$ for all $n > 2$.

### Theorem

Any functor $F : \text{CStar}^{\text{op}} \to \textbf{Top}$ (or $\textbf{Sets}$) which assigns the Gelfand spectrum to a commutative C*-algebra must yield the empty space (or set) on $\mathbb{M}_n(\mathbb{C})$ for all $n > 2$.

Key ingredient of the proofs: the Kochen-Specker Theorem.
Reyes’ theorem for other structures?

We wondered: What kind of dualities are ruled out by (extensions of) Reyes’ theorem?

In particular, could there still be a duality with a certain kind of
- locales,
- toposes,
- quantales?

Perhaps the conclusion of Reyes’ theorem should be that the spectrum of $\mathbb{M}_n(\mathbb{C})$ is a pointless space!
Our contribution

1. We give a simple conceptual proof of Reyes’ proof (which will fit on one page).
2. This proof gives a uniform reason why Reyes’ theorem also holds for locales, toposes and quantales.
3. And the argument gives some hints as to where one should look for dualities that might exist.

To explain this proof, I will have to revisit the Kochen-Specker Theorem.
Kochen-Specker Theorem

Let $\mathcal{H}$ be a Hilbert space of dimension $> 2$. Let $\mathcal{P}(\mathcal{H})$ be the lattice of closed linear subspaces of $\mathcal{H}$ (the propositions of quantum logic).

$\mathcal{P}(\mathcal{H})$ is not a Boolean algebra (it is not distributive), but it comes equipped with an involution $(-)\perp$ satisfying the orthomodular law.

Kochen-Specker Theorem (weak form)

If $\mathcal{P}(\mathcal{H}) \rightarrow \mathbb{B}$ is a morphism of orthomodular lattices whose codomain is a Boolean algebra, then its codomain is the trivial Boolean algebra in which $0 = 1$. 
Kochen-Specker Theorem, cont.

Definition

A *partial Boolean algebra* consists of a set \( \mathbb{P} \) equipped with a reflexive and symmetric commeasurability relation \( \circ \), partial binary operations \( \land, \lor : \circ \to \mathbb{P} \), a unary operation \( \neg : \mathbb{P} \to \mathbb{P} \) and constants \( 0, 1 \in \mathbb{P} \), satisfying the following condition:

*If \( A \) is a commeasurable subset of \( \mathbb{P} \) (i.e., \( a \circ b \) for all \( a, b \in A \)) and it contains \( 0, 1 \) and is closed under \( \neg, \lor, \land \), then these operations give \( A \) the structure of a Boolean algebra.*

Example: \( \mathcal{P}(\mathcal{H}) \) with \( a \circ b \), if \( a \) and \( b \) are commeasurable (the subalgebra generated by \( \{a, b\} \) is Boolean).
**Definition**

A map \( f : P \to Q \) is a morphism of partial Boolean algebra, if it preserves the commensurability relation and the algebraic operations, in so far as they are defined.

**Kochen-Specker Theorem**

If \( P(\mathcal{H}) \to B \) is a morphism of partial Boolean algebras whose codomain is a (total) Boolean algebra, then its codomain is the trivial Boolean algebra in which \( 0 = 1 \).

(For more on partial Boolean algebras, see our: Noncommutativity as a colimit, arXiv:1003.3618. To appear in *Applied Categorical Structures*.)
Kochen-Specker viewed categorically

Let $\mathcal{C}(\mathcal{P}(\mathcal{H}))$ be the diagram of commutative (distributive, Boolean) subalgebras of $\mathcal{P}(\mathcal{H})$ and inclusions between them.

Kochen-Specker Theorem (categorical rephrasing)

In the category of Boolean algebras, the colimit of $\mathcal{C}(\mathcal{P}(\mathcal{H}))$ is the terminal object.

\[
\begin{array}{ccc}
C_i & \rightarrow & C_j \\
\downarrow & & \downarrow \\
\mathcal{P}(\mathcal{H}) & \rightarrow & \mathbb{B}
\end{array}
\]

$\Rightarrow \quad \mathbb{B} \cong \{0 = 1\}$
Kochen-Specker Theorem for C*-algebras

Using this version of the Kochen-Specker Theorem we can also state in a crisp form its version for (unital) C*-algebras:

Kochen-Specker Theorem (for C*-algebras)

For a C*-algebra $A$, let $C(A)$ be the diagram of commutative subalgebras of $A$ and embeddings between them. Then, in the category $\text{cCStar}$, the colimit of $C(\mathbb{M}_n(\mathbb{C}))$ is the terminal object if $n > 2$.

The same is true for rings.
Alternative proof of Reyes’ theorem

Recall that Reyes’ theorem for C*-algebras was:

**Theorem**

Any functor $F : \text{CStar}^{\text{op}} \to \text{Top}$ (or $\text{Sets}$) which assigns the Gelfand spectrum to a commutative C*-algebra must yield the empty space (or set) on $\mathbb{M}_n(\mathbb{C})$ for all $n > 2$.

Our alternative proof relies on:

**Theorem**

The functor $\text{Gelf} : \text{cCstar}^{\text{op}} \to \text{Top}$ preserves limits.

**Proof**

Because compact Hausdorff spaces are closed under limits in $\text{Top}$. 
Proof of Reyes’ theorem

- Let $G : \text{CStar}^{\text{op}} \to \text{Top}$ be the functor which on a C*-algebra $A$ applies the Gelfand spectrum to all objects and maps in $\mathcal{C}(A)$ and then takes the limit of the resulting diagram in $\text{Top}$.
- $G$ assigns the Gelfand spectrum to every commutative C*-algebra.
- It is the final functor with this property.
- One can also compute $G(A)$ by first taking the colimit of $\mathcal{C}(A)$ in the category of commutative C*-algebras and then applying the Gelfand spectrum.
- Hence $G(\mathbb{M}_n(\mathbb{C})) = \emptyset$ for all $n > 2$.
- So if $F : \text{CStar}^{\text{op}} \to \text{Top}$ is any functor assigning the Gelfand spectrum to commutative C*-algebras, then there are maps $F(\mathbb{M}_n(\mathbb{C})) \to G(\mathbb{M}_n(\mathbb{C}))$. Hence also $F(\mathbb{M}_n(\mathbb{C})) = \emptyset$ for all $n > 2$. 
Reyes’ theorem for locales

Now it is clear that Reyes’ theorem applies to locales as well, because:

**Theorem**

Compact Hausdorff locales (compact completely regular locales) are closed under limits in the category $\textbf{Loc}$ of locales.

Hence:

**Theorem**

Any functor $F : \textbf{CStar}^{\text{op}} \to \textbf{Loc}$ which assigns the Gelfand locale to a commutative $C^*$-algebra must yield the trivial locale (in which $0=1$) on $\mathbb{M}_n(\mathbb{C})$ for all $n > 2$.

We have similar no-go theorems for toposes, ringed toposes, schemes.
Reyes’ theorem for quantales

Rather surprisingly, we also have:

**Theorem**

Compact Hausdorff locales (compact completely regular locales) are closed under limits in the category *Quant* of quantales.

Hence we have a no-go theorem for quantales as well.
Other spectra

**Theorem**

Any functor $F : \text{Ring}^{\text{op}} \to \text{Loc}$ which assigns the Zariski locale to a commutative ring must yield the trivial locale on $\mathbb{M}_n(\mathbb{C})$ for all $n > 2$.

It follows from this that any such functor $F$ must be trivial on all matrix rings $\mathbb{M}_n(R)$ for all rings $R$ which allow a homomorphism $\mathbb{C} \to R$.

Our proof is slightly different this time: here we reduce to the original result by Reyes.

One has similar no-go theorems for Stone and Peirce spectra.
So where to look for spectra

What can one still do? Different options are still available:

1. Assign to every C*-algebra $A$ a structure which is such that: if $A$ is commutative, then one may reconstruct the Gelfand spectrum from it. (E.g., the quantale of closed linear subspaces or the Bohrification.)

2. Invent a new notion of space in which the compact Hausdorff spaces are not closed under limits. (E.g., Rosický’s quantum frames.)

3. Change the morphisms.

So not all hope for a duality is lost. But considerable creativity will be required!